The function families used for the Lyness-Kaganove test are

$$\int_0^1 |x-\lambda|^\alpha \,\mathrm{d}x, \qquad \lambda\in[0,1],\ \alpha\in[-0.5,0] \tag{1}$$

$$\int_0^1 (x>\lambda)e^{\alpha x} \,\mathrm{d}x, \qquad \lambda\in[0,1],\ \alpha\in[0,1] \tag{2}$$

$$\int_0^1 \exp(-\alpha|x-\lambda|) \,\mathrm{d}x, \qquad \lambda\in[0,1],\ \alpha\in[0,4] \tag{3}$$

$$\int_1^2 10^\alpha/((x-\lambda)^2+10^\alpha) \,\mathrm{d}x, \qquad \lambda\in[1,2], \alpha\in[-6,-3] \tag{4}$$

$$\int_1^2 \sum_{i=1}^4 10^\alpha/((x-\lambda_i)^2+10^\alpha) \,\mathrm{d}x, \qquad \lambda_i\in[1,2], \alpha\in[-5,-3] \tag{5}$$

$$\int_0^1 2\beta(x-\lambda)\cos(\beta(x-\lambda)^2) \,\mathrm{d}x, \qquad \lambda\in[0,1],\ \alpha\in[1.8,2], \quad \beta = 10^\alpha/\max\{\lambda^2,(1-\lambda)^2\} \tag{6}$$

where the boolean expressions are evaluated to 0 or 1. The integrals are computed to relative precisions of $\tau = 10^{-3}$, 10^{-6}, 10^{-9} and 10^{-12} for 1 000 realizations of the random parameters λ and α. The results of these tests are shown in Table 1. For each function, the number of correct and incorrect integrations is given with, in brackets, the number of cases each where a warning (either explicit or whenever an error estimate larger than the requested tolerance is returned) was issued.

The functions used for the "battery" test are

$$
\begin{aligned}
f_1 &= \textstyle\int_0^1 e^x \,\mathrm{d}x & f_{14} &= \textstyle\int_0^{10} \sqrt{50}e^{-50\pi x^2} \,\mathrm{d}x \\
f_2 &= \textstyle\int_0^1 (x>0.3) \,\mathrm{d}x & f_{15} &= \textstyle\int_0^{10} 25e^{-25x} \,\mathrm{d}x \\
f_3 &= \textstyle\int_0^1 x^{1/2} \,\mathrm{d}x & f_{16} &= \textstyle\int_0^{10} 50(\pi(2500x^2+1))^{-1} \,\mathrm{d}x \\
f_4 &= \textstyle\int_{-1}^1 (\frac{23}{25}\cosh(x)-\cos(x)) \,\mathrm{d}x & f_{17} &= \textstyle\int_0^1 50(\sin(50\pi x)/(50\pi x))^2 \,\mathrm{d}x \\
f_5 &= \textstyle\int_{-1}^1 (x^4+x^2+0.9)^{-1} \,\mathrm{d}x & f_{18} &= \textstyle\int_0^\pi \cos(\cos(x)+3\sin(x)+2\cos(2x)+3\cos(3x)) \,\mathrm{d}x \\
f_6 &= \textstyle\int_0^1 x^{3/2} \,\mathrm{d}x & f_{19} &= \textstyle\int_0^1 \log(x) \,\mathrm{d}x \\
f_7 &= \textstyle\int_0^1 x^{-1/2} \,\mathrm{d}x & f_{20} &= \textstyle\int_{-1}^1 (1.005+x^2)^{-1} \,\mathrm{d}x \\
f_8 &= \textstyle\int_0^1 (1+x^4)^{-1} \,\mathrm{d}x & f_{21} &= \textstyle\int_0^1 \sum_{i=1}^3 \left[\cosh(20^i(x-2i/10))\right]^{-1} \,\mathrm{d}x \\
f_9 &= \textstyle\int_0^1 2(2+\sin(10\pi x))^{-1} \,\mathrm{d}x & f_{22} &= \textstyle\int_0^1 4\pi^2 x\sin(20\pi x)\cos(2\pi x) \,\mathrm{d}x \\
f_{10} &= \textstyle\int_0^1 (1+x)^{-1} \,\mathrm{d}x & f_{23} &= \textstyle\int_0^1 (1+(230x-30)^2)^{-1} \,\mathrm{d}x \\
f_{11} &= \textstyle\int_0^1 (1+e^x)^{-1} \,\mathrm{d}x & f_{24} &= \textstyle\int_0^3 \lfloor e^x \rfloor \,\mathrm{d}x \\
f_{12} &= \textstyle\int_0^1 x(e^x-1)^{-1} \,\mathrm{d}x & f_{25} &= \textstyle\int_0^5 (x+1)(x<1)+(3-x)(1\le x\le 3) \\
f_{13} &= \textstyle\int_0^1 \sin(100\pi x)/(\pi x) \,\mathrm{d}x & &\quad +2(x>3) \,\mathrm{d}x
\end{aligned}
$$

where the boolean expressions in f_2 and f_{25} evaluate to 0 or 1. The functions are taken from [1] with the following modifications:

- No special treatment is given to the case $x = 0$ in f_{12}, allowing the integrand to return NaN.
- f_{13} and f_{17} are integrated from 0 to 1 as opposed to 0.1 to 1 and 0.01 to 1 respectively, allowing the integrand to return NaN for $x = 0$.
- No special treatment of $x < 10^{-15}$ in f_{19} allowing the integrand to return −Inf.
- f_{24} was suggested by J. Waldvogel as a simple yet tricky test function with multiple discontinuities.
- f_{25} was introduced in [1], yet not used in the battery test therein.

References

[1] W. Gander and W. Gautschi, *Adaptive quadrature — revisited*, Tech. Rep. 306, Department of Computer Science, ETH Zurich, Switzerland, 1998.

$\tau = 10^{-3}$	quadl			DQAGS			da2glob			cquad			quadgk		
$f(x)$	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}
Eqn (1)	406	594	95.13	926	74	454.99	977	23	95.97	1000	0	280.15	454	546	238.23
Eqn (2)	894	106	117.15	959	41	398.12	1000	0	59.62	1000 (1)	0	175.34	993 (1)	7	267.27
Eqn (3)	868	132	45.39	998	2	179.51	1000	0	29.14	1000	0	112.72	1000 (1)	0	150.00
Eqn (4)	359	641	89.67	782	218 (49)	482.33	882	118	136.50	1000	0	342.12	719 (1)	281	404.46
Eqn (5)	303	697	308.76	969	31 (4)	1692.22	997	3	480.64	997	3	976.35	917 (1)	83 (1)	1300.89
Eqn (6)	994	6	746.19	1000	0	446.42	1000	0	523.35	1000	0	873.80	1000 (1)	0	491.85

$\tau = 10^{-6}$	quadl			DQAGS			da2glob			cquad			quadgk		
$f(x)$	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}
Eqn (1)	398	602	368.70	908	92	1093.43	993 (7)	7	298.45	1000	0	866.68	305 (1)	695	894.90
Eqn (2)	884	116	236.25	912	88	788.68	998	2	100.12	998	2	315.75	954 (1)	46	552.24
Eqn (3)	770	230	103.41	985	15	365.65	1000	0	58.12	1000	0	313.69	899 (1)	101	240.09
Eqn (4)	992	8	481.80	925 (1)	75 (75)	700.81	1000	0	290.52	1000	0	614.82	1000 (1)	0	788.43
Eqn (5)	982	18	1297.56	998	2 (2)	2031.20	1000	0	870.00	1000	0	1811.84	1000 (3)	0	2073.30
Eqn (6)	998	2 (2)	2029.44	1000	0	577.21	1000 (1)	0	695.02	1000	0	1196.39	1000 (1)	0	909.45

$\tau = 10^{-9}$	quadl			DQAGS			da2glob			cquad			quadgk		
$f(x)$	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}
Eqn (1)	367	633 (31)	1051.22	748 (68)	252 (151)	1784.79	874 (222)	126 (125)	627.40	910 (26)	90 (88)	1849.60	245 (22)	755 (406)	9987.36
Eqn (2)	877	123	356.58	860	140 (4)	1167.47	990	10	141.82	990	10	460.60	905 (1)	95	825.66
Eqn (3)	780	220	184.98	979	21	567.29	1000	0	93.74	1000	0	520.52	877 (1)	123	383.31
Eqn (4)	999	1	1211.85	924	76 (76)	824.29	1000	0	474.81	1000	0	1077.16	1000 (2)	0	1332.18
Eqn (5)	998	2	3336.06	998	2 (2)	2449.86	1000	0	1431.70	1000	0	3254.49	1000 (5)	0	3352.65
Eqn (6)	959	41 (41)	5186.43	1000 (2)	0	737.44	998 (63)	2 (2)	799.86	1000 (2)	0	1393.02	1000 (2)	0	1599.09

$\tau = 10^{-12}$	quadl			DQAGS			da2glob			cquad			quadgk		
$f(x)$	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}
Eqn (1)	261	739 (375)	2843.43	473 (110)	527 (493)	2632.14	494 (267)	506 (506)	1068.39	568 (103)	432 (432)	8593.15	62 (14)	938 (893)	17602.26
Eqn (2)	871	129	488.10	819 (4)	181 (23)	1533.59	990	10	184.42	990	10	606.36	804 (51)	196 (55)	1078.92
Eqn (3)	795	205	315.78	962	38	766.96	1000	0	133.60	1000	0	736.10	845 (1)	155	525.15
Eqn (4)	928	72	3217.08	866	134 (76)	995.06	996 (765)	4 (4)	710.62	1000 (426)	0	18000.49	888 (207)	112 (112)	7074.69
Eqn (5)	741	259 (259)	8697.84	998	2 (2)	2988.68	1000 (849)	0	2068.38	1000 (326)	0	12835.03	1000 (36)	0	8274.12
Eqn (6)	249	751 (751)	9601.47	962 (546)	38 (38)	1213.21	548 (548)	452 (452)	894.86	994 (474)	6 (6)	20098.05	981 (74)	19 (19)	4322.58

Table 1: Results of the Lyness-Kaganove tests for $\tau = 10^{-3}, 10^{-6}, 10^{-9}$ and 10^{-12}. The columns marked with ✓ and ✗ indicate the number of correct and incorrect results respectively, out of 1 000 runs. The numbers in brackets indicate the number of runs in which a warning was issued. The column n_{eval} contains the average number of function evaluations required for each run.

	$\tau = 10^{-3}$					$\tau = 10^{-6}$					$\tau = 10^{-9}$					$\tau = 10^{-12}$				
$f(x)$	quadl	DQAGS	da2glob	cquad	quadgk	quadl	DQAGS	da2glob	cquad	quadgk	quadl	DQAGS	da2glob	cquad	quadgk	quadl	DQAGS	da2glob	cquad	quadgk
f_1	18	21	**9**	33	150	18	21	**9**	33	150	18	21	**9**	33	150	18	21	**17**	33	150
f_2	108	357	**61**	161	240	198	357	**101**	301	540	318	357	**141**	441	840	408	357	**181**	581	1170
f_3	48	105	**25**	101	150	108	231	**65**	429	150	258	231	**137**	799	150	~~618~~	231	241	1191	**150**
f_4	18	21	**9**	33	150	18	21	**9**	33	150	**18**	21	25	33	150	48	**21**	33	33	150
f_5	18	21	**17**	33	150	48	**21**	33	95	150	**48**	63	65	95	150	168	**63**	65	219	150
f_6	18	21	**9**	33	150	48	105	**41**	159	150	108	189	**73**	359	150	288	189	**137**	607	150
f_7	289	231	**121**	269	150	439	231	285	709	**150**	~~889~~	231	581	1409	**150**	~~2429~~	231	965	2179	**150**
f_8	18	21	**17**	33	150	**18**	21	25	33	150	48	**21**	33	95	150	138	63	**49**	95	150
f_9	198	315	**121**	261	150	468	399	**233**	587	300	1038	567	**401**	991	570	2808	735	**577**	1425	1200
f_{10}	18	21	**9**	33	150	18	21	**17**	33	150	48	21	**17**	33	150	48	**21**	33	33	150
f_{11}	18	21	**9**	33	150	18	21	**9**	33	150	18	21	**9**	33	150	48	21	**17**	33	150
f_{12}	19	21	**9**	47	150	19	21	**9**	55	150	19	21	**9**	63	150	19	21	**9**	63	150
f_{13}	~~589~~	**651**	929	1403	780	1519	**1323**	1469	2347	1500	4879	**1323**	1913	2459	1950	~~10039~~	**1323**	2233	2521	3780
f_{14}	78	231	**45**	151	150	138	231	**65**	183	180	228	273	**105**	225	300	588	273	**153**	369	450
f_{15}	78	147	**41**	135	150	168	189	**69**	159	180	288	189	**101**	191	210	708	231	**145**	277	300
f_{16}	18	21	**9**	33	150	18	21	**9**	33	150	18	21	**9**	33	150	18	21	**9**	33	150
f_{17}	**79**	483	325	903	330	~~949~~	**777**	1065	1491	840	2839	**1323**	1725	2419	1740	6469	**1323**	2077	2451	3120
f_{18}	108	105	**73**	145	150	228	147	**129**	209	150	738	189	**185**	395	240	1758	**273**	**273**	581	390
f_{19}	109	231	**65**	255	150	229	231	**145**	717	210	~~499~~	**231**	285	1323	360	1369	**231**	449	1943	510
f_{20}	18	21	**17**	33	150	48	**21**	33	33	150	**48**	63	65	95	150	168	**63**	65	219	150
f_{21}	~~138~~	~~273~~	~~85~~	~~203~~	~~240~~	~~348~~	~~357~~	~~185~~	~~391~~	~~270~~	**1158**	~~441~~	~~273~~	~~653~~	~~510~~	2748	~~525~~	**649**	1839	~~870~~
f_{22}	228	**147**	241	371	150	888	315	**305**	627	330	2508	**315**	385	627	660	5568	**315**	513	627	930
f_{23}	108	273	**93**	191	210	258	399	**161**	365	360	588	441	**241**	569	510	1608	483	**401**	957	840
f_{24}	~~138~~	**1911**	~~453~~	4515	~~1380~~	~~1878~~	~~8211~~	~~857~~	**11433**	~~7290~~	~~3738~~	~~12285~~	~~1301~~	**18503**	~~12690~~	~~5538~~	~~16359~~	~~1745~~	**25191**	~~17010~~
f_{25}	108	567	**81**	277	300	348	819	**149**	593	840	528	819	**201**	933	1350	678	819	**269**	1253	1950

Table 2: Results of battery test for $\tau = 10^{-3}, 10^{-6}, 10^{-9}$ and 10^{-12}. The columns contain the number of function evaluations required by each integrator for each tolerance. For each test and tolerance, the best result (least function evaluations) is in bold and unsuccessful runs are stricken through.

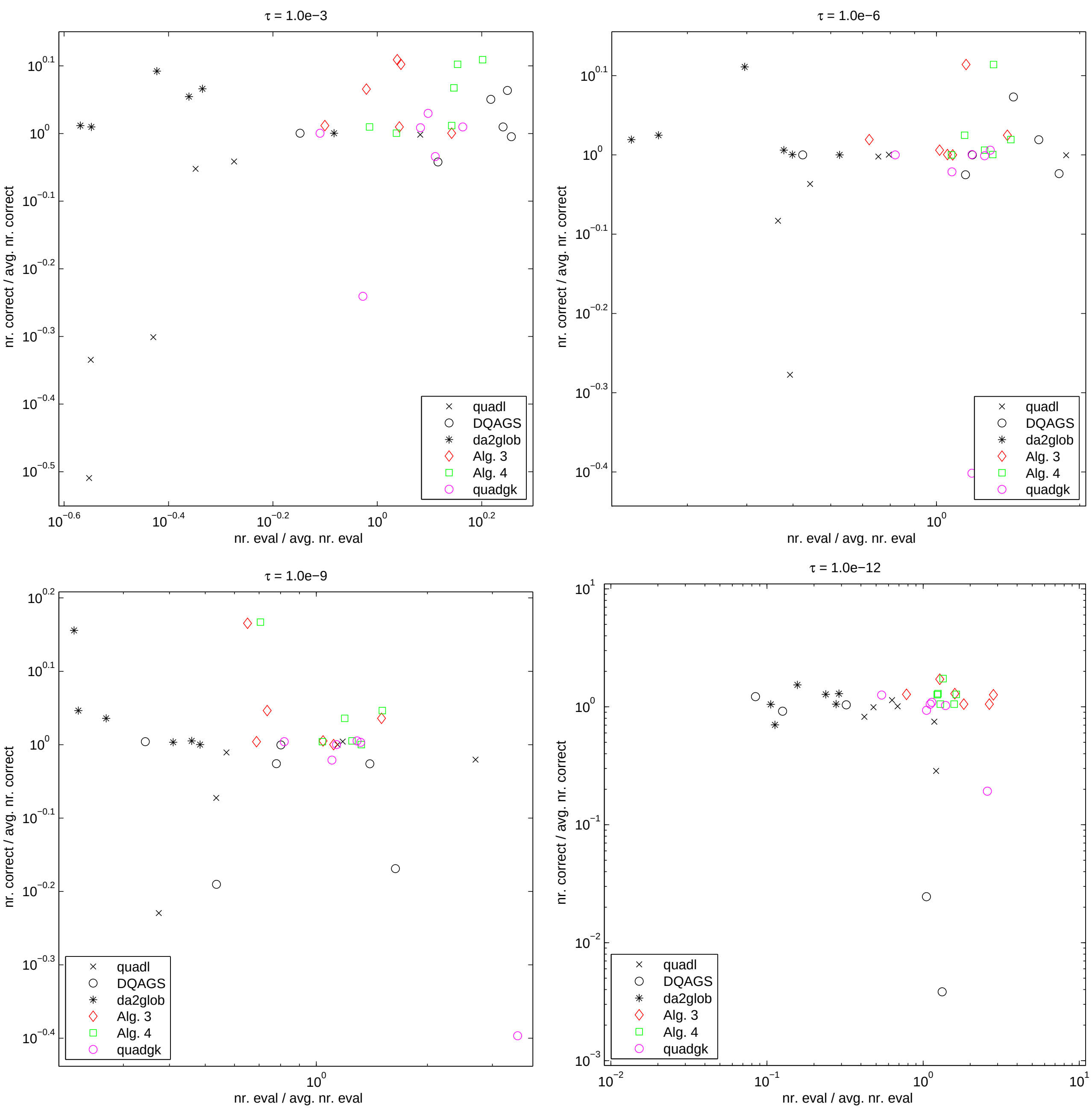

Figure 1: Scatter-plots of the results of the Lyness-Kaganove testsuite for each tolerance. Each point represents one of the test functions (Equations 1 to 6). Its location is determined by the relative number of function evaluations (on the x-axis) and the relative number of correct evaluations (on the y-axis).