The function families used for the Lyness-Kaganove test are

$$\int_0^1 |x-\lambda|^\alpha \,\mathrm{d}x, \qquad \lambda \in [0,1],\ \alpha \in [-0.5, 0] \tag{1}$$

$$\int_0^1 (x>\lambda) e^{\alpha x} \,\mathrm{d}x, \qquad \lambda \in [0,1],\ \alpha \in [0, 1] \tag{2}$$

$$\int_0^1 \exp(-\alpha|x-\lambda|) \,\mathrm{d}x, \qquad \lambda \in [0,1],\ \alpha \in [0, 4] \tag{3}$$

$$\int_1^2 10^\alpha/((x-\lambda)^2+10^\alpha) \,\mathrm{d}x, \qquad \lambda \in [1,2], \alpha \in [-6, -3] \tag{4}$$

$$\int_1^2 \sum_{i=1}^4 10^\alpha/((x-\lambda_i)^2+10^\alpha) \,\mathrm{d}x, \qquad \lambda_i \in [1,2], \alpha \in [-5, -3] \tag{5}$$

$$\int_0^1 2\beta(x-\lambda)\cos(\beta(x-\lambda)^2) \,\mathrm{d}x, \qquad \lambda \in [0,1],\ \alpha \in [1.8, 2], \tag{6}$$

$$\beta = 10^\alpha / \max\{\lambda^2, (1-\lambda)^2\}$$

where the boolean expressions are evaluated to 0 or 1. The integrals are computed to relative precisions of $\tau = 10^{-3}$, 10^{-6}, 10^{-9} and 10^{-12} for 100 realizations of the random parameters λ and α. The results of these tests are shown in Table 1. For each function, the number of correct and incorrect integrations is given with, in brackets, the number of cases each where a warning (either explicit or whenever an error estimate larger than the requested tolerance is returned) was issued.

The functions used for the "battery" test are

$$\begin{aligned}
f_1 &= \int_0^1 e^x \,\mathrm{d}x & f_{14} &= \int_0^{10} \sqrt{50} e^{-50\pi x^2} \,\mathrm{d}x \\
f_2 &= \int_0^1 (x>0.3) \,\mathrm{d}x & f_{15} &= \int_0^{10} 25 e^{-25x} \,\mathrm{d}x \\
f_3 &= \int_0^1 x^{1/2} \,\mathrm{d}x & f_{16} &= \int_0^{10} 50(\pi(2500x^2+1))^{-1} \,\mathrm{d}x \\
f_4 &= \int_{-1}^1 (\tfrac{23}{25}\cosh(x) - \cos(x)) \,\mathrm{d}x & f_{17} &= \int_0^1 50(\sin(50\pi x)/(50\pi x))^2 \,\mathrm{d}x \\
f_5 &= \int_{-1}^1 (x^4+x^2+0.9)^{-1} \,\mathrm{d}x & f_{18} &= \int_0^\pi \cos(\cos(x)+3\sin(x)+2\cos(2x)+3\cos(3x)) \,\mathrm{d}x \\
f_6 &= \int_0^1 x^{3/2} \,\mathrm{d}x & f_{19} &= \int_0^1 \log(x) \,\mathrm{d}x \\
f_7 &= \int_0^1 x^{-1/2} \,\mathrm{d}x & f_{20} &= \int_{-1}^1 (1.005+x^2)^{-1} \,\mathrm{d}x \\
f_8 &= \int_0^1 (1+x^4)^{-1} \,\mathrm{d}x & f_{21} &= \int_0^1 \sum_{i=1}^3 \left[\cosh(20^i(x-2i/10))\right]^{-1} \,\mathrm{d}x \\
f_9 &= \int_0^1 2(2+\sin(10\pi x))^{-1} \,\mathrm{d}x & f_{22} &= \int_0^1 4\pi^2 x \sin(20\pi x)\cos(2\pi x) \,\mathrm{d}x \\
f_{10} &= \int_0^1 (1+x)^{-1} \,\mathrm{d}x & f_{23} &= \int_0^1 (1+(230x-30)^2)^{-1} \,\mathrm{d}x \\
f_{11} &= \int_0^1 (1+e^x)^{-1} \,\mathrm{d}x & f_{24} &= \int_0^3 \lfloor e^x \rfloor \,\mathrm{d}x \\
f_{12} &= \int_0^1 x(e^x-1)^{-1} \,\mathrm{d}x & f_{25} &= \int_0^5 (x+1)(x<1) + (3-x)(1\le x\le 3) \\
f_{13} &= \int_0^1 \sin(100\pi x)/(\pi x) \,\mathrm{d}x & & \quad +2(x>3) \,\mathrm{d}x
\end{aligned}$$

where the boolean expressions in f_2 and f_{25} evaluate to 0 or 1. The functions are taken from [?] with the following modifications:

- No special treatment is given to the case $x = 0$ in f_{12}, allowing the integrand to return NaN.
- f_{13} and f_{17} are integrated from 0 to 1 as opposed to 0.1 to 1 and 0.01 to 1 respectively, allowing the integrand to return NaN for $x = 0$.
- No special treatment of $x < 10^{-15}$ in f_{19} allowing the integrand to return $-$Inf.
- f_{24} was suggested by J. Waldvogel as a simple yet tricky test function with multiple discontinuities.
- f_{25} was introduced in [?], yet not used in the battery test therein.

$\tau = 10^{-3}$	quadl			DQAGS			da2glob			cquad			quadgk		
$f(x)$	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}
Eqn (1)	44 (0)	56 (0)	95.70	91 (0)	9 (0)	440.58	99 (0)	1 (0)	95.40	100 (0)	0 (0)	280.00	40 (0)	60 (0)	249.00
Eqn (2)	91 (0)	9 (0)	115.80	97 (0)	3 (0)	398.16	100 (0)	0 (0)	58.76	100 (0)	0 (0)	173.90	98 (0)	2 (0)	292.20
Eqn (3)	86 (0)	14 (0)	43.50	100 (0)	0 (0)	178.50	100 (0)	0 (0)	28.72	100 (0)	0 (0)	110.58	100 (0)	0 (0)	150.00
Eqn (4)	39 (0)	61 (0)	95.70	74 (0)	26 (2)	427.14	89 (0)	11 (0)	137.96	100 (0)	0 (0)	334.60	74 (0)	26 (0)	443.70
Eqn (5)	33 (0)	67 (0)	317.40	96 (0)	4 (0)	1702.26	100 (0)	0 (0)	482.96	100 (0)	0 (0)	963.94	91 (1)	9 (0)	1437.00
Eqn (6)	100 (0)	0 (0)	794.40	100 (0)	0 (0)	462.42	100 (0)	0 (0)	534.68	100 (0)	0 (0)	891.94	100 (0)	0 (0)	529.80

$\tau = 10^{-6}$	quadl			DQAGS			da2glob			cquad			quadgk		
$f(x)$	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}
Eqn (1)	41 (0)	59 (0)	356.40	90 (0)	10 (0)	1084.86	98 (0)	2 (0)	293.80	100 (0)	0 (0)	869.12	33 (0)	67 (0)	1037.40
Eqn (2)	91 (0)	9 (0)	235.80	94 (0)	6 (0)	788.76	100 (0)	0 (0)	99.20	100 (0)	0 (0)	314.34	95 (0)	5 (0)	575.70
Eqn (3)	76 (0)	24 (0)	102.90	99 (0)	1 (0)	366.66	100 (0)	0 (0)	58.56	100 (0)	0 (0)	313.00	91 (0)	9 (0)	263.10
Eqn (4)	100 (0)	0 (0)	485.70	94 (0)	6 (6)	697.20	100 (0)	0 (0)	289.80	100 (0)	0 (0)	614.74	100 (0)	0 (0)	835.50
Eqn (5)	100 (0)	0 (0)	1302.30	100 (0)	0 (0)	2029.86	100 (0)	0 (0)	874.88	100 (0)	0 (0)	1816.58	100 (1)	0 (0)	2220.00
Eqn (6)	100 (0)	0 (0)	2168.40	100 (0)	0 (0)	592.62	100 (0)	0 (0)	707.96	100 (0)	0 (0)	1207.74	100 (0)	0 (0)	966.60

$\tau = 10^{-9}$	quadl			DQAGS			da2glob			cquad			quadgk		
$f(x)$	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}
Eqn (1)	30 (0)	70 (3)	1037.10	44 (43)	56 (55)	1510.32	88 (18)	12 (11)	618.84	87 (8)	13 (13)	1825.78	21 (0)	79 (41)	9435.90
Eqn (2)	89 (0)	11 (0)	356.40	65 (64)	35 (26)	1037.82	100 (0)	0 (0)	141.12	100 (0)	0 (0)	460.04	90 (0)	10 (0)	847.80
Eqn (3)	74 (0)	26 (0)	182.10	89 (72)	11 (9)	484.68	100 (0)	0 (0)	94.48	100 (0)	0 (0)	518.58	88 (0)	12 (0)	413.70
Eqn (4)	100 (0)	0 (0)	1200.60	94 (50)	6 (6)	808.50	100 (0)	0 (0)	472.12	100 (0)	0 (0)	1074.12	100 (1)	0 (0)	1545.90
Eqn (5)	100 (0)	0 (0)	3358.80	100 (95)	0 (0)	2338.14	100 (0)	0 (0)	1438.04	100 (0)	0 (0)	3276.90	100 (0)	0 (0)	3569.70
Eqn (6)	94 (0)	6 (6)	5427.60	100 (88)	0 (0)	679.14	100 (7)	0 (0)	819.56	100 (7)	0 (0)	1335.22	98 (5)	2 (2)	1902.30

$\tau = 10^{-12}$	quadl			DQAGS			da2glob			cquad			quadgk		
$f(x)$	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}	✓	✗	n_{eval}
Eqn (1)	24 (0)	76 (34)	2771.80	0 (0)	100 (100)	1510.32	52 (26)	48 (46)	1060.60	52 (9)	48 (48)	8531.26	3 (0)	97 (93)	17457.30
Eqn (2)	90 (0)	10 (0)	489.90	1 (1)	99 (91)	1037.82	100 (0)	0 (0)	183.36	100 (0)	0 (0)	603.84	85 (0)	15 (5)	1705.80
Eqn (3)	76 (0)	24 (0)	310.80	4 (2)	96 (94)	484.68	100 (0)	0 (0)	134.48	100 (0)	0 (0)	732.74	90 (0)	10 (0)	556.50
Eqn (4)	94 (0)	6 (0)	3211.80	88 (86)	12 (12)	845.88	100 (72)	0 (0)	695.96	100 (85)	0 (0)	14741.66	67 (43)	33 (33)	8437.20
Eqn (5)	73 (0)	27 (26)	8747.10	99 (99)	1 (1)	2362.92	100 (90)	0 (0)	2073.88	100 (91)	0 (0)	9380.36	88 (78)	12 (12)	11877.00
Eqn (6)	20 (0)	80 (79)	9696.30	95 (95)	5 (5)	679.14	57 (57)	43 (43)	915.56	98 (98)	2 (2)	18171.02	2 (2)	98 (98)	15574.50

Table 1: Results of the Lyness-Kaganove tests for $\tau = 10^{-3}, 10^{-6}, 10^{-9}$ and 10^{-12}. The columns marked with ✓ and ✗ indicate the number of correct and incorrect results respectively, out of 1 000 runs. The numbers in brackets indicate the number of runs in which a warning was issued. The column n_{eval} contains the average number of function evaluations required for each run.

$\tau = 10^{-3}$	quadl	DQAGS	da2glob	cquad	quadgk
Eqn (1)	12.98 (135.62)	21.13 (47.96)	59.95 (628.39)	3.59 (12.83)	8.04 (32.27)
Eqn (2)	16.20 (139.93)	20.51 (51.52)	42.12 (716.74)	3.02 (17.38)	11.09 (37.95)
Eqn (3)	5.71 (131.24)	12.30 (68.89)	17.05 (593.76)	1.25 (11.27)	3.52 (23.45)
Eqn (4)	12.60 (131.64)	12.69 (29.72)	79.00 (572.60)	3.41 (10.19)	14.49 (32.66)
Eqn (5)	49.70 (156.60)	163.63 (96.13)	286.53 (593.28)	32.24 (33.45)	19.60 (13.64)
Eqn (6)	114.05 (143.57)	25.71 (55.59)	263.66 (493.13)	12.88 (14.44)	6.65 (12.56)

$\tau = 10^{-6}$	quadl	DQAGS	da2glob	cquad	quadgk
Eqn (1)	49.57 (139.08)	51.16 (47.16)	156.65 (533.19)	9.98 (11.48)	27.99 (26.98)
Eqn (2)	33.13 (140.51)	40.09 (50.83)	72.73 (733.13)	5.83 (18.54)	24.41 (42.40)
Eqn (3)	14.36 (139.59)	24.58 (67.03)	35.89 (612.80)	3.94 (12.57)	9.14 (34.73)
Eqn (4)	66.33 (136.56)	20.42 (29.29)	137.79 (475.48)	4.95 (8.05)	21.16 (25.33)
Eqn (5)	204.74 (157.21)	194.24 (95.69)	429.41 (490.82)	46.22 (25.44)	22.01 (9.91)
Eqn (6)	312.34 (144.04)	32.83 (55.39)	317.51 (448.48)	14.52 (12.03)	7.91 (8.19)

$\tau = 10^{-9}$	quadl	DQAGS	da2glob	cquad	quadgk
Eqn (1)	147.57 (142.29)	72.56 (48.04)	290.44 (469.32)	19.04 (10.43)	45.13 (4.78)
Eqn (2)	50.74 (142.36)	53.00 (51.07)	105.07 (744.55)	8.58 (18.65)	38.03 (44.86)
Eqn (3)	26.16 (143.67)	32.39 (66.83)	56.00 (592.74)	6.65 (12.82)	16.70 (40.37)
Eqn (4)	165.19 (137.59)	23.70 (29.31)	198.13 (419.66)	7.45 (6.93)	22.86 (14.79)
Eqn (5)	525.70 (156.51)	222.79 (95.28)	616.81 (428.92)	71.12 (21.70)	24.09 (6.75)
Eqn (6)	779.36 (143.59)	37.44 (55.13)	354.00 (431.94)	15.76 (11.80)	9.37 (4.93)

$\tau = 10^{-12}$	quadl	DQAGS	da2glob	cquad	quadgk
Eqn (1)	390.94 (141.04)	71.51 (47.35)	432.62 (407.91)	84.44 (9.90)	43.19 (2.47)
Eqn (2)	69.85 (142.58)	53.04 (51.11)	136.99 (747.11)	11.70 (19.38)	49.53 (29.04)
Eqn (3)	44.30 (142.53)	31.94 (65.90)	75.80 (563.68)	9.30 (12.70)	23.42 (42.08)
Eqn (4)	439.79 (136.93)	24.68 (29.17)	256.14 (368.04)	107.28 (7.28)	26.32 (3.12)
Eqn (5)	1372.87 (156.95)	226.07 (95.67)	787.34 (379.64)	200.74 (21.40)	30.47 (2.57)
Eqn (6)	1404.06 (144.80)	38.08 (56.07)	390.08 (426.06)	219.67 (12.09)	18.38 (1.18)

Table 2: Timings of the results of the Lyness-Kaganove tests for $\tau = 10^{-3}, 10^{-6}, 10^{-9}$ and 10^{-12}. The values in each column are the average number of millisecond per integration and the values in brackets are the average number of microseconds per evaluated point of the integrand.

$f(x)$	$\tau = 10^{-3}$					$\tau = 10^{-6}$					$\tau = 10^{-9}$					$\tau = 10^{-12}$				
	quadl	DQAGS	da2glob	cquad	quadgk	quadl	DQAGS	da2glob	cquad	quadgk	quadl	DQAGS	da2glob	cquad	quadgk	quadl	DQAGS	da2glob	cquad	quadgk
f_1	18	21	**9**	33	150	18	21	**9**	33	150	18	21	**9**	33	150	18	21	**17**	33	150
f_2	108	357	**61**	161	270	198	357	**101**	301	570	318	357	**141**	441	870	408	357	**181**	581	1200
f_3	48	105	**25**	101	150	108	231	**65**	429	150	258	231	**137**	799	150	~~618~~	231	241	1191	**150**
f_4	18	21	**9**	33	150	18	21	**9**	33	150	**18**	21	25	33	150	48	**21**	33	33	150
f_5	18	21	**17**	33	150	48	**21**	33	95	150	**48**	63	65	95	150	168	**63**	65	219	150
f_6	18	21	**9**	33	150	48	105	**41**	159	150	108	189	**73**	359	150	288	~~189~~	**137**	607	150
f_7	289	231	**121**	253	150	439	231	285	693	**150**	~~889~~	231	581	1403	**150**	~~2429~~	231	965	2171	**150**
f_8	18	21	**17**	33	150	**18**	21	25	33	150	48	**21**	33	95	150	138	**21**	49	95	150
f_9	198	315	**121**	261	150	468	399	**233**	587	330	1038	483	**401**	991	570	2808	**483**	577	1425	1230
f_{10}	18	21	**9**	33	150	18	21	**17**	33	150	48	21	**17**	33	150	48	**21**	33	33	150
f_{11}	18	21	**9**	33	150	18	21	**9**	33	150	18	21	**9**	33	150	48	21	**17**	33	150
f_{12}	19	21	**9**	47	150	19	21	**9**	55	150	19	21	**9**	63	150	19	21	**9**	63	150
f_{13}	~~589~~	**651**	929	1399	780	1519	**1323**	1469	2347	1500	4879	**1323**	1913	2459	2640	~~10039~~	**1323**	2233	2521	4050
f_{14}	78	231	**45**	151	150	138	231	**65**	183	210	228	231	**105**	215	300	588	231	**153**	365	540
f_{15}	78	147	**41**	135	150	168	189	**69**	159	180	288	189	**101**	191	240	708	189	**145**	277	360
f_{16}	18	21	**9**	33	150	18	21	**9**	33	150	18	21	**9**	33	150	18	21	**9**	33	150
f_{17}	**79**	483	325	705	360	~~949~~	**777**	1065	1491	840	2839	**1197**	1725	2419	1740	6469	**1197**	2077	2451	3390
f_{18}	108	105	**73**	145	150	228	147	**129**	209	150	738	**147**	185	395	240	1758	**147**	273	581	450
f_{19}	109	231	**65**	285	150	229	231	**145**	717	240	~~499~~	**231**	285	1323	390	1369	**231**	449	1943	540
f_{20}	18	21	**17**	33	150	48	**21**	33	33	150	**48**	63	65	95	150	168	**63**	65	219	150
f_{21}	~~138~~	~~273~~	~~85~~	~~191~~	~~270~~	~~348~~	~~357~~	~~185~~	~~395~~	~~300~~	**1158**	~~357~~	~~273~~	~~653~~	~~570~~	2748	~~357~~	**649**	1839	~~900~~
f_{22}	228	**147**	241	371	150	888	315	**305**	627	330	2508	**315**	385	627	660	5568	**315**	513	627	1350
f_{23}	108	273	**93**	191	270	258	399	**161**	365	390	588	399	**241**	569	540	1608	**399**	401	957	1230
f_{24}	~~138~~	**1911**	~~453~~	4519	~~1740~~	~~1878~~	~~5061~~	~~857~~	**11389**	~~7830~~	~~3738~~	~~5061~~	~~1301~~	**18529**	~~13230~~	~~5538~~	~~5061~~	~~1745~~	**25237**	~~18570~~
f_{25}	108	567	**81**	277	330	348	819	**149**	593	870	528	819	**201**	933	1380	678	819	**269**	1253	2100

Table 3: Results of battery test for $\tau = 10^{-3}, 10^{-6}, 10^{-9}$ and 10^{-12}. The columns contain the number of function evaulations required by each integrator for each tolerance. For each test and tolerance, the best result (least function evaluations) is in bold and unsuccessful runs are stricken through.