MTH 253 – Calculus III December 2, 2005

Midterm 1

Name : _____ **Score :** (/190) = _____%

Problem 1. Does the sequence converge? If so, find it's limit.

a) $\left\{\frac{1}{5+2^{-n}}\right\}_{n=1}^{\infty}$ (/10)

Since $\lim_{n\to\infty}\frac{1}{5+2^{-n}}=\infty$ the sequence does not converge.

b) $a_n=\frac{n^2+1}{3n^2-3}$ (/10)

Since $\lim_{n\to\infty}\frac{n^2+1}{3n^2-3}=\frac{1}{3}$ the sequence converges to $\frac{1}{3}$.

c) $a_1=1 \quad a_{n+1}=(a_n)^2-3$ (/10)

Because $\{a_n\}=\{1,-2,1,-2,...\}$ the sequence does not converge.

Problem 2. What does it mean to say that $\sum_{n=1}^{\infty} a_n=9$? (/10)

The limit as $n\to\infty$ of the sequence of partial sums $S_n=\sum_{k=0}^{n} a_k$ is 9.

Problem 3. Does the series converge? If so, find its sum.

a) $\sum_{n=1}^{\infty} 5\cdot 10^{-n}$ (/10)

This is a *geometric series* $\sum_{n=1}^{\infty} a\,r^n$ with $a=5$ and $r=\frac{1}{10}$. Because $r<1$, the series is convergent.

$$5\cdot 10^{-n} = 5\cdot\left(\frac{1}{10}\right)^n$$

$$\sum_{n=0}^{\infty} a\,r^n = a+a\,r+a\,r^2+a\,r^3+\cdots = \frac{a}{1-r} \qquad \forall r\,|\,r<1$$

$$\sum_{n=1}^{\infty} a\,r^n = a\,r+a\,r^2+a\,r^3+\cdots = \left(\sum_{n=0}^{\infty} a\,r^n\right)-a$$

Let $a=5$ and $r=\frac{1}{10}$.

$$\sum_{n=1}^{\infty} 5\cdot\left(\frac{1}{10}\right)^n = \frac{5}{1-\frac{1}{10}}-5=\frac{5}{9}$$

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b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-1}$

(/10)

By comparison to a p-series $\frac{1}{n^p}$ with $p=1$, $\frac{1}{n} < \frac{1}{n-1}$, the series does not converge because a p-series is convergent only when $p>1$, and the terms of this series are always larger than the terms of $\frac{1}{n}$.

Problem 4. Identify each series as convergent or divergent. Tell which test you use to decide.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{n^2}$

(/10)

Since each term of the given series is always greater than the corresponding term of the p-series $\frac{2n}{n^2}$, it diverges by the *Comparison Test*.

$$\begin{aligned} \frac{2n+3}{n^2} &> \frac{2n}{n^2} \\ &> \frac{2}{n} \end{aligned}$$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}}{3^n}$

(/10)

Because $\lim_{n\to\infty} \frac{2^{2n}}{3^n} = \infty$, the series is divergent by the *Limit Test for Divergence*.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

(/10)

- This is an alternating series of the form $\sum (-1)^n b_n$ where $b_n = \frac{1}{n^{\frac{1}{3}}}$.
- $b_{n+1} \le b_n$ because $\frac{1}{(n+1)^{\frac{1}{3}}} < \frac{1}{n^{\frac{1}{3}}}$.
- $\lim_{n\to\infty} b_n = 0$

Because the series passes the above three tests, it is shown to be convergent by the *Alternating Series Test*. Note that it is **not** *absolutely convergent* because $\sum_{n=1}^{\infty} b_n$ does not converge, since it is a p-series with $p<1$.

Problem 5. Find the intervals of convergence for the following power series.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{n\,(x-1)^n}{3\cdot 2^n}$

(/10)

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(n+1)\,(x-1)^{n+1}}{3\cdot 2^{n+1}}\cdot\frac{3\cdot 2^n}{n\,(x-1)^n}\right| &= \lim_{n\to\infty}\left|\frac{n\,x+x-n-1}{2\,n}\right| \\
&= \left|\frac{x-1}{2}\right| \\
\frac{|x-1|}{2} &< 1 \\
|x-1| &< 2 \\
-2 < x-1 &< 2 \\
-1 < x &< 3 \\
\lim_{n\to\infty}\left(\frac{n\,(x-1)^n}{3\cdot 2^n}\right)_{x=-1} &= \lim_{n\to\infty}\frac{n\,(-2)^n}{3\,2^n}\notin\mathbb{R} \\
\lim_{n\to\infty}\left(\frac{n\,(x-1)^n}{3\cdot 2^n}\right)_{x=3} &= \lim_{n\to\infty}\frac{n}{3}=\infty
\end{aligned}$$

Therefore, the interval of convergence for this power series is $x\in(-1\,...\,3\,)$.

b) $\displaystyle\sum_{n=0}^{\infty} \frac{2\,x^n}{n!}$

(/10)

$$\lim_{n\to\infty}\left|\frac{2\,x^{n+1}}{(n+1)!}\cdot\frac{n!}{2\,x^n}\right| = \lim_{n\to\infty}\left|\frac{n!\,x}{(n+1)!}\right|=0$$

Therefore, the series is convergent for all x.

Problem 6. Find power series representations of the following functions:

a) $\dfrac{x}{2+4\,x}$

(/10)

$$\begin{aligned}
\text{Since} \quad \sum [f(x)\cdot g(x)] &= \sum f(x)\cdot \sum g(x) \\
\text{and} \quad \sum_{n=0}^{\infty} a\,r^n = a + a\,r + a\,r^2 + a\,r^3 + \cdots &= \frac{a}{1-r} \\
\text{so} \quad \frac{1}{1-x} &= x^0 + x^1 + x^2 + x^3 + \cdots \\
\text{and} \quad \frac{x}{2+4\,x} &= \frac{x}{2}\cdot\frac{1}{1-(-2\,x)} \\
\frac{x}{2+4\,x} &= \frac{x}{2}(-2\,x)^0 + \frac{x}{2}(-2\,x)^1 + \frac{x}{2}(-2\,x)^2 + \cdots \\
&= \frac{x}{2} - x^2 + 2\,x^3 - 4\,x^4 + \cdots
\end{aligned}$$

b) $\ln(1+x)$

(/15)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\ln(1+x) &= \frac{1}{1+x} = \frac{1}{1-(-x)} \\
&= \sum_{n=0}^{\infty} (-x)^n \\
\int \sum_{n=0}^{\infty} (-x)^n \,\mathrm{d}x &= \int (1 - x + x^2 - x^3 + \cdots)\,\mathrm{d}x \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}
\end{aligned}$$

Problem 7. Fill in the blank:

If $f(x)$ has a power series representation $f(x) = c_n\,(x-a)^n$ then $c_n =$ _____.

(/10)

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Problem 8. Find a power series for:

a) e^{3x}

(/10)

$$\begin{aligned}
f(x) = \mathrm{e}^{3x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,x^n \\
f^{(0)}(x) &= \mathrm{e}^{3x} \qquad f^{(0)}(0) = 1 \\
f^{(1)}(x) &= 3\,\mathrm{e}^{3x} \qquad f^{(1)}(0) = 3 \\
f^{(2)}(x) &= 9\,\mathrm{e}^{3x} \qquad f^{(2)}(0) = 9 \\
f^{(3)}(x) &= 27\,\mathrm{e}^{3x} \qquad f^{(3)}(0) = 27 \\
\mathrm{e}^{3x} &= 1 + 3\,x + \frac{9}{2}\,x^2 + \frac{9}{2}\,x^3 + \cdots
\end{aligned}$$

b) $e^x \ln(1+x)$ up to degree $= 3$. (Hint: #6b)

(/15)

$$\begin{aligned}
\text{Since} \quad \sum [f(x)\cdot g(x)] &= \sum f(x) \cdot \sum g(x) \\
e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\
e^x \cdot \ln(1+x) &= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) \\
&\approx \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) \\
&\approx x + \frac{x^2}{2} + \frac{x^3}{3}
\end{aligned}$$

Note that if the two expressions in the first approximation are not carried out to degree $= 3$, the final result will not be accurate, and the result may appear to be $x + \frac{x^2}{2} - \frac{x^4}{4}$.

Problem 9. Evaluate as a series:

$$\int_0^2 \sin x^2 \, dx$$

(/20)

$$\begin{aligned}
\sin x^2 &= x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} + \cdots \\
\int \sin x^2 \, dx &= \frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} + \cdots \\
\left.\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} + \cdots\right]_0^2 &= \left(\frac{2^3}{3} - \frac{2^7}{42} + \frac{2^{11}}{1320} + \cdots\right) - \left(\frac{0^3}{3} - \frac{0^7}{42} + \frac{0^{11}}{1320} + \cdots\right) \\
&= \left(\frac{2^3}{3} - \frac{2^7}{42} + \frac{2^{11}}{1320} + \cdots\right) \\
&\approx \frac{1352}{1155} \\
&\approx 1.17056
\end{aligned}$$