Improved fits for $E_1(x)$ *vis-à-vis* those presented in ARL-TR-1758

Efforts were made since the publication of ARL-TR-1758[1] to improve upon the accuracy of the fits presented in that document. Using the terminology of that report, the fit is actually done on $F_1(w)$, where $w = 1/x$ and the function $F_1 = E_1 e^{1/w}/w$ (*i.e.*, $F_1 = xe^x E_1$). This transformation provides for a function $F_1(w)$ which varies smoothly from unity (at $w = 0$) to zero (as $w \to \infty$). The fits were developed to minimize the largest occurance of the error function $\epsilon = [(F_1)_{\text{fit}} - F_1]/F_1$ at any point on the functional domain ($0 \le w < \infty$).

Several fits were developed, with greater accuracy resulting from the use of more parameters. In all cases, the fitted value of F_1 takes the form

$$(F_1)_{\text{fit}} = 1/w \cdot \ln\left\{1 + w - [w - \ln(1+w)] \cdot f(w)\right\} \quad . \tag{1}$$

Different fitting forms for f are presented below, with the associated values of $\max(\epsilon)$ and the value of w for which the $\max(\epsilon)$ occurs.

6-Parameter Fit:

$$f(w) = \frac{1 + Aw + (1 - e^{-\gamma})Bw^2}{1 + \left(A + \frac{5}{3}\right) w \frac{1 + C_1 w^2}{1 + D_1 w^2} + Bw^2 \frac{C_2 + w^2}{D_2 + w^2}} \quad , \tag{2}$$

where

$A = 4.69041102625857590\text{D}{+}00$
$B = 7.76097664114015200\text{D}{+}00$
$C_1 = 2.30659438867190370\text{D}{-}01$
$D_1 = 2.37827207448962350\text{D}{-}01$
$C_2 = 1.40614716699937220\text{D}{+}02$
$D_2 = 1.39838508752857140\text{D}{+}02$

$\max(\epsilon) = 1.709092\text{E}{-}05$ at $w = 99.782947$.

8-Parameter Fit:

$$f(w) = \frac{1 + Aw + (1 - e^{-\gamma})Bw^2}{1 + \left(A + \frac{5}{3}\right) w + Bw^2 \left(1 + \frac{C_1}{D_1 + w} + \frac{C_2 w}{(D_2 + w)^2} + \frac{C_3 w^2}{(D_3 + w)^3}\right)} \quad , \tag{3}$$

where

$A = 4.99191263403800090\text{D}{+}00$
$B = 8.55392302479002180\text{D}{+}00$
$C_1 = 2.41723456633978670\text{D}{+}00$
$D_1 = 3.38951420119681250\text{D}{+}02$
$C_2 = -1.22812127267525170\text{D}{+}00$
$D_2 = 1.37651629589202030\text{D}{+}02$
$C_3 = -5.35044576792352150\text{D}{-}02$
$D_3 = 8.67574715486645330\text{D}{-}01$

$\max(\epsilon) = 3.256059\text{E}{-}06$ at $w = 14.116516$.

10-Parameter Fit:

$$f(w) = \frac{1 + Aw + B(1 - e^{-\gamma})w^2}{1 + \left(A + \frac{5}{3}\right) w \frac{1 + E_1 w^2 + C_1 w^3}{1 + G_1 w^2 + D_1 w^3} + Bw^2 \frac{C_2 + E_2 w + w^2}{D_2 + G_2 w + w^2}} \quad , \tag{4}$$

where

$A = 7.74823271875959920\text{D}{+}00$
$B = 2.42331702915707580\text{D}{+}01$
$C_1 = 3.48159341103108960\text{D}{-}03$
$D_1 - 1.01421240602993130\text{D}{-}03$
$C_2 = 7.77813337451483160\text{D}{+}01$
$D_2 = 9.41171735762913930\text{D}{+}01$
$E_1 = 9.52872674659433150\text{D}{-}01$
$G_1 = 9.43128313648529780\text{D}{-}01$
$E_2 = 6.29952753733901360\text{D}{+}01$
$G_2 = 6.28722150790988370\text{D}{+}01$

$\max(\epsilon) = 1.882788\text{E}{-}06$ at $w = 45.275091$.

12-Parameter Fit:

$$f(w) = \frac{1 + Aw + B(1-e^{-\gamma})w^2}{1 + \dfrac{A + 5/3 + H_1 w}{1 + H_2 w}\, w\, \dfrac{1 + E_1 w^2 + C_1 w^3}{1 + G_1 w^2 + D_1 w^3} + Bw^2 \dfrac{C_2 + E_2 w + w^2}{D_2 + G_2 w + w^2}} \quad , \tag{5}$$

where

$A = 6.58272149564893640\mathrm{D}{+}00$
$B = 1.65978816181933130\mathrm{D}{+}01$
$C_1 = 5.14357696611253590\mathrm{D}{-}03$
$D_1 = 2.33075091216811960\mathrm{D}{-}03$
$C_2 = 7.13737271086793040\mathrm{D}{+}01$
$D_2 = 8.03410183545183260\mathrm{D}{+}01$
$E_1 = 7.58135938490494650\mathrm{D}{-}01$
$G_1 = 7.56175417133594260\mathrm{D}{-}01$
$E_2 = 4.63752805419194370\mathrm{D}{+}01$
$G_2 = 4.63079462234788510\mathrm{D}{+}01$
$H_1 = 2.97036738105914930\mathrm{D}{-}03$
$H_2 = 1.92452599275716720\mathrm{D}{-}04$

$\max(\epsilon) = 3.076770\mathrm{E}{-}07$ at $w = 0.072472$.

References

1. S. B. Segletes. A compact analytical fit to the exponential integral $E_1(x)$. Technical Report ARL-TR-1758, U.S. Army Ballistic Research Laboratory, Aberdeen Proving Ground, MD, September 1998.

Steven B. Segletes
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