

The function families used for the Lyness-Kaganove test are

$$\int_0^1 |x - \lambda|^\alpha dx, \quad \lambda \in [0, 1], \alpha \in [-0.5, 0] \quad (1)$$

$$\int_0^1 (x > \lambda)e^{\alpha x} dx, \quad \lambda \in [0, 1], \alpha \in [0, 1] \quad (2)$$

$$\int_0^1 \exp(-\alpha|x - \lambda|) dx, \quad \lambda \in [0, 1], \alpha \in [0, 4] \quad (3)$$

$$\int_1^2 10^\alpha / ((x - \lambda)^2 + 10^\alpha) dx, \quad \lambda \in [1, 2], \alpha \in [-6, -3] \quad (4)$$

$$\int_1^2 \sum_{i=1}^4 10^\alpha / ((x - \lambda_i)^2 + 10^\alpha) dx, \quad \lambda_i \in [1, 2], \alpha \in [-5, -3] \quad (5)$$

$$\int_0^1 2\beta(x - \lambda) \cos(\beta(x - \lambda)^2) dx, \quad \lambda \in [0, 1], \alpha \in [1.8, 2], \quad (6)$$

$$\beta = 10^\alpha / \max\{\lambda^2, (1 - \lambda)^2\}$$

where the boolean expressions are evaluated to 0 or 1. The integrals are computed to relative precisions of $\tau = 10^{-3}$, 10^{-6} , 10^{-9} and 10^{-12} for 100 realizations of the random parameters λ and α . The results of these tests are shown in Table 1. For each function, the number of correct and incorrect integrations is given with, in brackets, the number of cases each where a warning (either explicit or whenever an error estimate larger than the requested tolerance is returned) was issued.

The functions used for the ‘‘battery’’ test are

$$\begin{aligned} f_1 &= \int_0^1 e^x dx & f_{14} &= \int_0^{10} \sqrt{50}e^{-50\pi x^2} dx \\ f_2 &= \int_0^1 (x > 0.3) dx & f_{15} &= \int_0^{10} 25e^{-25x} dx \\ f_3 &= \int_0^1 x^{1/2} dx & f_{16} &= \int_0^{10} 50(\pi(2500x^2 + 1))^{-1} dx \\ f_4 &= \int_{-1}^1 (\frac{23}{25} \cosh(x) - \cos(x)) dx & f_{17} &= \int_0^1 50(\sin(50\pi x)/(50\pi x))^2 dx \\ f_5 &= \int_{-1}^1 (x^4 + x^2 + 0.9)^{-1} dx & f_{18} &= \int_0^\pi \cos(\cos(x) + 3\sin(x) + 2\cos(2x) + 3\cos(3x)) dx \\ f_6 &= \int_0^1 x^{3/2} dx & f_{19} &= \int_0^1 \log(x) dx \\ f_7 &= \int_0^1 x^{-1/2} dx & f_{20} &= \int_{-1}^1 (1.005 + x^2)^{-1} dx \\ f_8 &= \int_0^1 (1 + x^4)^{-1} dx & f_{21} &= \int_0^1 \sum_{i=1}^3 [\cosh(20^i(x - 2i/10))]^{-1} dx \\ f_9 &= \int_0^1 2(2 + \sin(10\pi x))^{-1} dx & f_{22} &= \int_0^1 4\pi^2 x \sin(20\pi x) \cos(2\pi x) dx \\ f_{10} &= \int_0^1 (1 + x)^{-1} dx & f_{23} &= \int_0^1 (1 + (230x - 30)^2)^{-1} dx \\ f_{11} &= \int_0^1 (1 + e^x)^{-1} dx & f_{24} &= \int_0^3 [e^x] dx \\ f_{12} &= \int_0^1 x(e^x - 1)^{-1} dx & f_{25} &= \int_0^5 (x + 1)(x < 1) + (3 - x)(1 \leq x \leq 3) \\ f_{13} &= \int_0^1 \sin(100\pi x)/(\pi x) dx & & + 2(x > 3) dx \end{aligned}$$

where the boolean expressions in f_2 and f_{25} evaluate to 0 or 1. The functions are taken from [?] with the following modifications:

- No special treatment is given to the case $x = 0$ in f_{12} , allowing the integrand to return NaN.
- f_{13} and f_{17} are integrated from 0 to 1 as opposed to 0.1 to 1 and 0.01 to 1 respectively, allowing the integrand to return NaN for $x = 0$.
- No special treatment of $x < 10^{-15}$ in f_{19} allowing the integrand to return $-\text{Inf}$.
- f_{24} was suggested by J. Waldvogel as a simple yet tricky test function with multiple discontinuities.
- f_{25} was introduced in [?], yet not used in the battery test therein.

$\tau = 10^{-3}$		quad1		DQAGS		da2g1lob		cquad		quadgk	
$f(x)$	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓
Eqn (1)	44 (0)	56 (0)	95.70	91 (0)	440.58	99 (0)	1 (0)	95.40	100 (0)	282.98	40 (0)
Eqn (2)	91 (0)	9 (0)	115.80	97 (0)	398.16	100 (0)	0 (0)	58.76	100 (0)	175.02	98 (0)
Eqn (3)	86 (0)	14 (0)	43.50	100 (0)	178.50	100 (0)	0 (0)	28.72	100 (0)	110.58	100 (0)
Eqn (4)	39 (0)	61 (0)	95.70	74 (0)	427.14	89 (0)	11 (0)	137.96	100 (0)	341.00	74 (0)
Eqn (5)	33 (0)	67 (0)	317.40	96 (0)	1702.26	100 (0)	0 (0)	482.96	100 (0)	981.40	91 (1)
Eqn (6)	100 (0)	0 (0)	794.40	100 (0)	462.42	100 (0)	0 (0)	534.68	100 (0)	891.94	100 (0)
$\tau = 10^{-6}$		quad1		DQAGS		da2g1lob		cquad		quadgk	
$f(x)$	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓
Eqn (1)	41 (0)	59 (0)	356.40	90 (0)	1084.86	98 (0)	2 (0)	293.80	100 (0)	872.38	33 (0)
Eqn (2)	91 (0)	9 (0)	235.80	94 (0)	788.76	100 (0)	0 (0)	99.20	100 (0)	314.32	95 (0)
Eqn (3)	76 (0)	24 (0)	102.90	99 (0)	366.66	100 (0)	0 (0)	58.56	100 (0)	313.00	91 (0)
Eqn (4)	100 (0)	0 (0)	485.70	94 (0)	697.20	100 (0)	0 (0)	289.80	100 (0)	614.52	100 (0)
Eqn (5)	100 (0)	0 (0)	1302.30	100 (0)	2029.86	100 (0)	0 (0)	874.88	100 (0)	1815.78	100 (1)
Eqn (6)	100 (0)	0 (0)	2168.40	100 (0)	592.62	100 (0)	0 (0)	707.96	100 (0)	1207.74	100 (0)
$\tau = 10^{-9}$		quad1		DQAGS		da2g1lob		cquad		quadgk	
$f(x)$	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓
Eqn (1)	30 (0)	70 (3)	1037.10	44 (43)	56 (55)	88 (18)	12 (11)	618.84	90 (1)	1841.94	21 (0)
Eqn (2)	89 (0)	11 (0)	356.40	65 (64)	35 (26)	1037.82	0 (0)	141.12	100 (0)	459.84	90 (0)
Eqn (3)	74 (0)	26 (0)	182.10	89 (72)	11 (9)	484.68	0 (0)	94.48	100 (0)	518.58	88 (0)
Eqn (4)	100 (0)	0 (0)	1200.60	94 (50)	6 (6)	808.50	0 (0)	1074.12	100 (0)	1074.12	100 (1)
Eqn (5)	100 (0)	0 (0)	3358.80	100 (95)	0 (0)	2338.14	0 (0)	1438.04	100 (0)	3276.90	100 (0)
Eqn (6)	94 (0)	6 (6)	5427.60	100 (88)	0 (0)	679.14	0 (0)	819.56	100 (7)	1335.22	98 (5)
$\tau = 10^{-12}$		quad1		DQAGS		da2g1lob		cquad		quadgk	
$f(x)$	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓	n_{eval}	✓
Eqn (1)	24 (0)	76 (35)	2771.80	0 (0)	100 (100)	52 (26)	48 (46)	1060.60	58 (6)	8203.76	3 (0)
Eqn (2)	90 (0)	10 (0)	489.90	1 (1)	99 (91)	1037.82	0 (0)	183.36	100 (0)	604.24	85 (0)
Eqn (3)	76 (0)	24 (0)	310.80	4 (2)	96 (94)	484.68	0 (0)	134.48	100 (0)	732.74	90 (0)
Eqn (4)	94 (0)	6 (0)	3211.80	88 (86)	12 (12)	845.88	0 (0)	695.96	100 (81)	17623.62	67 (43)
Eqn (5)	73 (0)	27 (26)	8747.10	99 (99)	1 (1)	2362.92	0 (0)	2073.88	100 (91)	13337.56	88 (78)
Eqn (6)	20 (0)	80 (79)	9696.30	95 (95)	5 (5)	679.14	43 (43)	915.56	98 (98)	23682.62	2 (2)

Table 1: Results of the Lyness-Kaganove tests for $\tau = 10^{-3}, 10^{-6}, 10^{-9}$ and 10^{-12} . The columns marked with ✓ and X indicate the number of correct and incorrect results respectively, out of 1000 runs. The numbers in brackets indicate the number of runs in which a warning was issued. The column n_{eval} contains the average number of function evaluations required for each run.

$\tau = 10^{-3}$	quadl	DQAGS	da2g1ob	cquad	quadgk
Eqn (1)	13.26 (138.58)	21.62 (49.07)	61.67 (646.47)	120.01 (424.08)	8.54 (34.31)
Eqn (2)	16.31 (140.81)	20.83 (52.32)	42.82 (728.79)	86.52 (494.33)	10.71 (36.65)
Eqn (3)	5.85 (134.39)	12.33 (69.05)	17.53 (610.33)	34.41 (311.16)	3.63 (24.21)
Eqn (4)	12.98 (135.64)	13.03 (30.52)	81.46 (590.43)	158.40 (464.52)	14.86 (33.50)
Eqn (5)	50.27 (158.39)	164.31 (96.53)	295.16 (611.16)	516.21 (526.00)	19.92 (13.86)
Eqn (6)	116.63 (146.82)	26.37 (57.02)	273.52 (511.56)	374.55 (419.92)	6.85 (12.92)

$\tau = 10^{-6}$	quadl	DQAGS	da2g1ob	cquad	quadgk
Eqn (1)	51.06 (143.28)	52.58 (48.46)	163.80 (557.51)	349.25 (400.34)	28.72 (27.68)
Eqn (2)	33.68 (142.84)	40.67 (51.56)	74.90 (755.08)	164.30 (522.70)	24.67 (42.85)
Eqn (3)	14.64 (142.28)	24.73 (67.45)	36.51 (623.41)	105.33 (336.52)	9.31 (35.37)
Eqn (4)	67.69 (139.37)	20.96 (30.07)	142.34 (491.16)	233.34 (379.72)	21.59 (25.84)
Eqn (5)	209.59 (160.94)	196.99 (97.04)	448.47 (512.60)	794.84 (437.74)	22.73 (10.24)
Eqn (6)	320.45 (147.78)	33.77 (56.99)	332.36 (469.46)	440.59 (364.80)	8.47 (8.76)

$\tau = 10^{-9}$	quadl	DQAGS	da2g1ob	cquad	quadgk
Eqn (1)	148.90 (143.57)	72.75 (48.17)	297.52 (480.77)	704.19 (382.31)	45.48 (4.82)
Eqn (2)	51.16 (143.55)	53.29 (51.35)	107.16 (759.38)	245.52 (533.93)	38.08 (44.92)
Eqn (3)	26.31 (144.50)	32.46 (66.98)	57.33 (606.77)	179.19 (345.53)	16.78 (40.57)
Eqn (4)	167.97 (139.90)	24.21 (29.94)	204.17 (432.45)	362.16 (337.17)	23.29 (15.07)
Eqn (5)	535.29 (159.37)	224.23 (95.90)	638.27 (443.85)	1304.35 (398.04)	24.53 (6.87)
Eqn (6)	792.67 (146.05)	38.23 (56.30)	365.66 (446.17)	474.49 (355.36)	9.58 (5.04)

$\tau = 10^{-12}$	quadl	DQAGS	da2g1ob	cquad	quadgk
Eqn (1)	400.51 (144.49)	72.98 (48.32)	437.10 (412.13)	4131.01 (503.55)	43.55 (2.49)
Eqn (2)	71.62 (146.20)	54.02 (52.05)	138.39 (754.76)	324.12 (536.41)	50.30 (29.49)
Eqn (3)	46.15 (148.48)	33.17 (68.44)	77.83 (578.73)	255.74 (349.02)	24.30 (43.67)
Eqn (4)	450.01 (140.11)	25.33 (29.94)	258.57 (371.53)	10687.76 (606.45)	26.35 (3.12)
Eqn (5)	1395.41 (159.53)	227.43 (96.25)	790.61 (381.22)	7470.82 (560.13)	30.39 (2.56)
Eqn (6)	1420.25 (146.47)	38.47 (56.65)	390.22 (426.21)	14501.96 (612.35)	18.13 (1.16)

Table 2: Timings of the results of the Lyness-Kaganove tests for $\tau = 10^{-3}$, 10^{-6} , 10^{-9} and 10^{-12} . The values in each column are the average number of millisecond per integration and the values in brackets are the average number of microseconds per evaluated point of the integrand.

$f(x)$	$\tau = 10^{-3}$				$\tau = 10^{-6}$				$\tau = 10^{-9}$				$\tau = 10^{-12}$							
	quadl	DQAGS	da2glob	cquad	quadgk	quadl	DQAGS	da2glob	cquad	quadgk	quadl	DQAGS	da2glob	cquad	quadgk	quadl	DQAGS	da2glob	cquad	quadgk
f_1	18	21	9	33	150	18	21	9	33	150	18	21	9	33	150	18	21	17	33	150
f_2	108	357	61	161	270	198	357	101	301	570	318	357	141	441	870	408	357	181	581	1200
f_3	48	105	25	101	150	108	231	65	429	150	258	231	137	799	150	648	231	241	1191	150
f_4	18	21	9	33	150	18	21	9	33	150	18	21	9	33	150	48	21	33	33	150
f_5	18	21	17	33	150	48	21	33	95	150	48	63	65	95	150	168	63	65	219	150
f_6	18	21	9	33	150	48	105	41	159	150	108	189	73	359	150	288	489	137	607	150
f_7	289	231	121	269	150	439	231	285	709	150	889	231	581	1409	150	2429	231	965	2179	150
f_8	18	21	17	33	150	18	21	25	33	150	48	21	33	95	150	138	21	49	95	150
f_9	198	315	121	261	150	468	399	233	587	330	1038	483	401	991	570	2808	483	577	1425	1230
f_{10}	18	21	9	33	150	18	21	17	33	150	18	21	17	33	150	48	21	33	33	150
f_{11}	18	21	9	33	150	18	21	9	33	150	18	21	9	33	150	48	21	17	33	150
f_{12}	19	21	9	47	150	19	21	9	55	150	19	21	9	63	150	19	21	9	63	150
f_{13}	589	651	929	1403	780	1519	1323	1469	2347	1500	4879	1323	1913	2459	2640	49039	1323	2233	2521	4050
f_{14}	78	231	45	151	150	138	231	65	183	210	228	231	105	225	300	588	231	153	369	540
f_{15}	78	147	41	135	150	168	189	69	159	180	288	189	101	191	240	708	189	145	277	360
f_{16}	18	21	9	33	150	18	21	9	33	150	18	21	9	33	150	18	21	9	33	150
f_{17}	79	483	325	903	360	949	777	1065	1491	840	2839	1197	1725	2419	1740	6469	1197	2077	2451	3390
f_{18}	108	105	73	145	150	228	147	129	209	150	738	147	185	395	240	1758	147	273	581	450
f_{19}	109	231	65	255	150	229	231	145	717	240	499	231	285	1323	390	1369	231	449	1943	540
f_{20}	18	21	17	33	150	48	21	33	33	150	48	63	65	95	150	168	63	65	219	150
f_{21}	438	273	85	203	270	348	357	485	391	300	1158	357	273	653	570	2748	357	649	1839	900
f_{22}	228	147	241	371	150	888	315	305	627	330	2508	315	385	627	660	5568	315	513	627	1350
f_{23}	108	273	93	191	270	258	399	161	365	390	588	399	241	569	540	1608	399	401	957	1230
f_{24}	438	1911	453	4515	1740	4878	5064	857	11433	7830	3738	5064	4301	18503	43230	5538	5064	4745	25199	48570
f_{25}	108	567	81	277	330	348	819	149	593	870	528	819	201	933	1380	678	819	269	1253	2100

Table 3: Results of battery test for $\tau = 10^{-3}, 10^{-6}, 10^{-9}$ and 10^{-12} . The columns contain the number of function evaluations required by each integrator for each tolerance. For each test and tolerance, the best result (least function evaluations) is in bold and unsuccessful runs are stricken through.