1 Claim

The bin-wise average of M DFTs of consecutive, non-overlapping input samples is the same as the DFT of the sample-wise averaged, consecutive, non-overlapping input vectors.

2 Proof

As mention in the claim, we average over M DFTs. Let the length of the individual DFT be N.

Let us consider a complex input signal $x[n]$, with n being the non-negative index, that exists for the whole observation, hence $n \in \{0, 1, \dots, MN\}$.

Lets use the same definition of the discrete Fourier transform as is used in FFTW [1], so that the first DFT X_1 would yield in bin k:

$$X_{1,k} = \sum_{n=0}^{N-1} x_n e^{-j2\pi k \frac{n}{N}} . \tag{1}$$

Introducing the average DFT Y, we see that

$$\begin{aligned} Y_k &= \frac{1}{M} \sum_{m=0}^{M-1} X_{m,k} && (2) \\ &= \frac{1}{M} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x_{n+Nm} e^{-j2\pi k \frac{n}{N}} . && (3) \end{aligned}$$

With the sums above being finite, we can change their order:

$$= \frac{1}{M} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x_{n+Nm} e^{-j2\pi k \frac{n}{N}} \tag{4}$$

The exponential term doesn't depend on m, and hence can be extracted from the inner sum.

$$\begin{aligned} &= \sum_{n=0}^{N-1} e^{-j2\pi k \frac{n}{N}} \frac{1}{M} \sum_{m=0}^{M-1} x_{n+Nm} && (5) \\ &= \sum_{n=0}^{N-1} e^{-j2\pi k \frac{n}{N}} x_{\text{avg}} && (6) \end{aligned}$$

Notice that

$$x_{\text{avg}} = \frac{1}{M} \sum_{m=0}^{M-1} x_{n+Nm} \tag{7}$$

is the sample average over M consecutive, non-overlapping sample vectors of length N.

■

3 SNR considerations

Let us consider the job of the DFT to find the coefficients that you'd have to write in front of the individual series representing an N long complex oscillation with frequency $\frac{1}{N}k$, i.e. the set of series

$$s_k = \left(e^{-j2\pi k \frac{n}{N}}\right)_{n=0,\ldots,N-1} . \tag{8}$$

All the $\frac{1}{\frac{1}{N}k} = \frac{N}{k}$-periodic oscillations are also N-periodic (by definition of periodicity).

Hence, in consecutive vectors of length N, if you add up the elements with the same index, and the observed signal is periodic, you simply get the averaging factor M as the factor between the individual vector and the sum of M vectors.

Now, remember that the purpose of averaging the DFTs was to enhance SNR by M. That's exactly what happens when you average the input signal, too.

References

[1] FFTW, the Fastest Fourier Transform in the West, *What FFTW Really Computes: The 1d Discrete Fourier Transform (DFT)*, `http://www.fftw.org/doc/The-1d-Discrete-Fourier-Transform-_0028DFT_0029.html#The-1d-Discrete-Fourier-Transform-_0028DFT_0029`